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Fraternal augmentations, arrangeability and linear Ramsey numbers

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ABSTRACT

We relate the notions of arrangeability and admissibility to bounded expansion classes and prove that these notions can be characterized by $\nabla_1(G)$. (The Burr–Erdős conjecture relates to $\nabla_0(G)$.) This implies the linearity of the Ramsey number and the bounded game chromatic number for some new classes of graphs.

1. Introduction

Ramsey theory is a domain of the theory of (very) large numbers; see e.g. [10,16]. However there are exceptions: among them are game versions of Ramsey problems (see [2]) and the detailed analysis of the (generalized) Ramsey number, defined for an arbitrary graph G as the least integer $r(G)$, the *Ramsey number of* G, such that for every graph H of order at least $r(G)$ either H or its complement contains G as a subgraph. When the graph G is sparse, then we can expect small Ramsey numbers (and in many cases exact results). Such results often belong more to graph theory than to Ramsey theory. But this is not the case with the linear Ramsey numbers where the analysis involves techniques from the very heart of Ramsey theory.

1.1. Linearly bounded Ramsey numbers

In this paper we will consider only simple loopless graphs. Let G be a graph. It follows from Ramsey's theorem that there exists a least integer $r(G)$, the *Ramsey number of* G, such that for every graph H of order at least $r(G)$ either H or its complement contains G as a subgraph. A family of graphs $\mathcal{F}$ is a *Ramsey linear family* if there exists a constant $c = c(\mathcal{F})$ such that $r(G) \leq cn$ for every $G \in \mathcal{F}$ of order n. In 1973, Burr and Erdős [3] formulated the following conjecture.

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Conjecture 1. *For each positive integer p, there exists a constant c_p such that if G is a p-degenerate graph on n vertices then $r(G) < c_p n$.*

In 1983 Chvátal, Rödl, Szemerédi, and Trotter [5] proved that the conjecture holds for graphs with bounded maximum degree (improved in [8]; tight bounds for the bipartite case appear in [9]). This result has been extended to p-arrangeable graphs by Chen and Schelp [4].

Recall that a graph G is *p-arrangeable* (a concept introduced in [4]) if its vertices can be ordered as $v_1, v_2, \ldots, v_n$ in such a way that $|N_{L_i}(N_{R_i}(v_i))| \leq p$ for each $1 \leq i \leq n$, where $L_i = \{v_1, v_2, \ldots, v_i\}$, $R_i = \{v_{i+1}, v_{i+2}, \ldots, v_n\}$, and $N_A(B)$ denotes the neighbors of B which lie in A.

In [4], the authors proved that planar graphs are p-arrangeable for some p. In [22], Rödl and Thomas prove that graphs with no subdivision of K_q are p-arrangeable for some p depending on q. The bound on the Ramsey number of p-arrangeable graphs was improved by Eaton [6] and then by Graham, Rödl and Ruciński [8].

The Burr–Erdős conjecture is known to hold for subdivided graphs [1] (improved in [15]). Moreover, some further progress toward the conjecture may also be found in [12–14]. A general survey of what is known on Ramsey numbers may be found in [21].

In this paper we will give a new sufficient condition for a graph to be p-arrangeable based on a new graph invariant, the *greatest reduced average degree (grad)* with *rank r* of a graph G, $\nabla_r(G)$. This invariant is defined by $\nabla_r(G) = \max \frac{|E(H)|}{|V(H)|}$, where the maximum is taken over all the simple minors H of G obtained by contracting a set of vertex-disjoint subgraphs with radius at most r and then deleting any number of edges and vertices (see [18,20,19] and Section 2). Obviously these invariants form a non-decreasing sequence: $\nabla_0(G) \leq \nabla_1(G) \leq \nabla_2(G) \leq \cdots$. Note that the bounded degeneracy is equivalent to bounded $\nabla(0)$. According to this definition, Conjecture 1 may be restated as:

Conjecture 2 (*Alternative Form of the Burr–Erdős Conjecture*). *There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that for any graph G of order n,*

$$\frac{r(G)}{n} < f(\nabla_0(G)).$$

The above results on linear Ramsey numbers relate to *bounded expansion classes*, that is classes $\mathcal{C}$ for which $\sup_{G\in\mathcal{C}} \nabla_r(G)$ is bounded (then $F(r) = \sup_{G\in\mathcal{C}} \nabla_r(G)$ is called the *expansion* of the class). For example, the class of planar graphs has its expansion uniformly bounded by 3 and the uniform bound for the expansion in fact characterizes minor closed classes; see [18]. The class of graphs with degrees bounded by d is also a bounded expansion class whose expansion is bounded by an exponential function in d. In Section 5 of this paper (see also [20]) we show that, for every k, the class of graphs not containing a topological subdivision of K_k is a bounded expansion class. In this paper we show that for any graph G its arrangeability can be actually bounded as a function of $\nabla_1(G)$ only. Combining this with [8] we obtain:

Theorem 1.1. *There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that for any graph G of order n,*

$$\frac{r(G)}{n} < f(\nabla_1(G)).$$

More precisely,

$$\log_2\left(\frac{r(G)}{n}\right) = O((\nabla_0(G)\nabla_1(G)\log\nabla_1(G))^2).$$

Also, combining our results with those of [23,24] we get:

Corollary 1.2. *There exists a function $g : \mathbb{R}^2 \to \mathbb{R}$ such that for any graphs G_1, G_2,*

$$\frac{r(G_1, G_2)}{\max(|V(G_1)|, |V(G_2)|)} \leq g(\nabla_0(G_1), \nabla_1(G_2)).$$

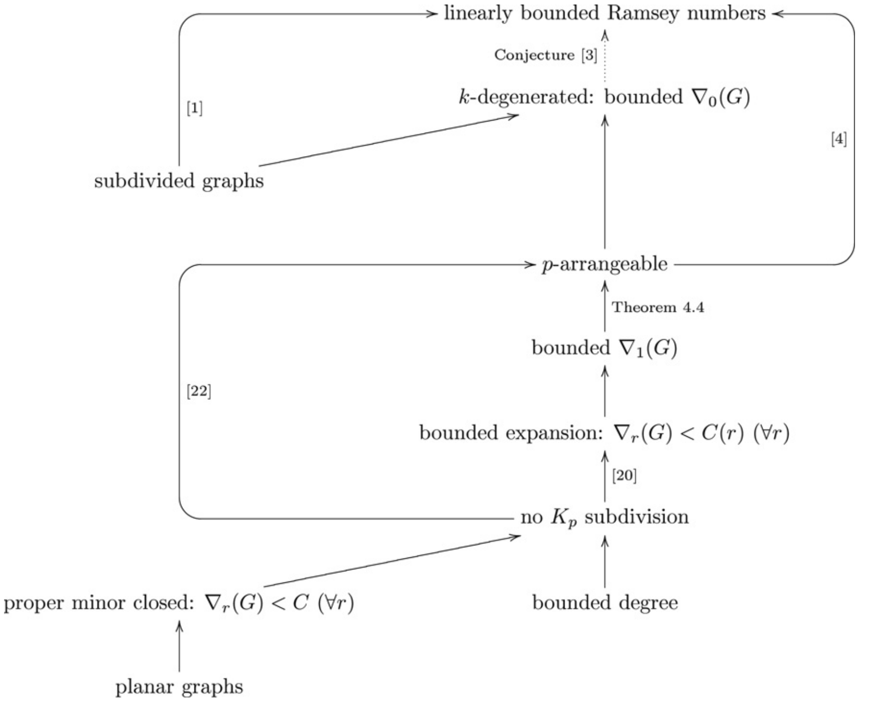

Fig. 1. Inclusion of graph classes.

2. Definitions and notation

2.1. Arrangeability

Let p be an integer. The class of p-arrangeable graphs is known to have linearly bounded Ramsey numbers [4]. Up to now, the better bounds are given by the following result (extending earlier results of Eaton [6]):

Theorem 2.1 (*Graham, Rödl, Ruciński [8]*). *For some positive constant c and all integers $p \geq 2$ and all $n \geq p + 1$, if H is a p-arrangeable graph with n vertices then*

$$\log_2 \frac{r(G)}{n} \leq cp(\log p)^2. \tag{1}$$

A related concept of *admissibility* was introduced by Kierstead and Trotter [11] in the context of *game chromatic numbers*.

Definition 2.1. Let G be a graph, let $M \subseteq V(G)$, and let $v \in M$. A set $A \subseteq V(G)$ is called an *M-blade* with center v if either

(1) $A = \{a\}$ and $a \in M$ is adjacent to v, or
(2) $A = \{a, b\}$, $a \in M - \{v\}$, $b \in V(G) - M$, and b is adjacent to both v and a.

An *M-fan* with center v is a set of pairwise disjoint M-blades with center v. Let k be an integer. A graph G is *k-admissible* if the vertices of G can be numbered $v_1, v_2, \ldots, v_n$ in such a way that for every $i = 1, 2, \ldots, n$, G has no $\{v_1, v_2, \ldots, v_i\}$-fan with center v_i of size $k + 1$.

The concepts of arrangeability and admissibility are almost equivalent (Fig. 1).

Lemma 2.2 (*Kierstead, Trotter [11]*). *Let k be an integer. Any k-arrangeable graph is 2k-admissible; any k-admissible graph is $(k^2 - k + 1)$-arrangeable.*

This result is central to the proof that graphs with no K_p subdivisions are $(\frac{1}{2}p^2(p^2+1))$-admissible [22]. We give here a new characterization of arrangeability in terms of fraternal augmentations. This in turn extends known classes of graphs for which the Burr–Erdős conjecture is valid.

2.2. The grad and class expansion

We review some definitions and notation introduced in [17–20]:

The *distance* $d(x, y)$ between two vertices x and y of a graph is the minimum length of a path linking x and y, or ∞ if x and y do not belong to the same connected component. The *radius* $\rho(G)$ of a connected graph G is the minimum maximum distance of the vertices from a fixed vertex, that is $\rho(G) = \min_{r\in V(G)} \max_{x\in V(G)} \mathrm{d}(r, x)$. The *radius* $\rho(G)$ of a non-connected graph G is the maximum of the radii of its components.

A (simple) graph H is a *minor* of a graph G if it may be obtained from G by contracting edges (and simplifying the resulting graph), deleting edges and deleting vertices. In such a case we write $H \preceq G$. As edge deletion and contraction commute, we may consider contractions first and deletions next. Thus a minor H of a graph G is obtained by contracting some connected subset F of edges, simplifying and then taking a subgraph (i.e. $H \subseteq G/F$). Notice that the subset F is in general not uniquely determined by G and H. We denote by G_F the subgraph of G induced by the subset F of edges of G. The *depth* of a minor of G is the minimum radius of the part that we have to contract in G to get H. More formally,

$$\mathrm{depth}(H, G) = \min\{\rho(G_F) : H \subseteq G/F\}.$$

Definition 2.2. The *greatest reduced average density (grad) of G with rank r* is

$$\nabla_r(G) = \max_{\substack{H\preceq G \\ \mathrm{depth}(H,G)\le r}} \frac{|E(H)|}{|V(H)|}.$$

The first grad, ∇_0, is closely related to the degeneracy or maximum average degree (G is k-degenerate iff $k \geq \lfloor 2\nabla_0(G)\rfloor$; note that none of the results of this paper hold for k-degenerate graphs; higher grads are needed – but rather surprisingly, ∇_1 will suffice). The grads ∇_r form a non-decreasing sequence which is eventually constant, starting from some index (smaller than the order of the graph).

2.3. Orientation and fraternal augmentation

A digraph $\vec{G}$ is *fraternally oriented* if $(x, z) \in E(\vec{G})$ and $(y, z) \in E(\vec{G})$ implies $(x, y) \in E(\vec{G})$ or $(y, x) \in E(\vec{G})$. This concept was introduced by Skrien [25] and a characterization of fraternally oriented digraphs having no symmetrical arcs has been obtained by Gavril and Urrutia [7], who also proved that triangulated graphs and circular arc graphs are all fraternally orientable graphs. An orientation is *transitive* if $(x, y) \in E(\vec{G})$ and $(y, z) \in E(\vec{G})$ implies $(x, z) \in E(\vec{G})$. It is obvious that a graph has an acyclic transitive fraternal orientation in which every vertex has indegree at most $(k - 1)$ if and only if it is the closure of a rooted forest of height k.

Definition 2.3. Let $\vec{G}$ be a directed graph.

A 1-*fraternal augmentation* of $\vec{G}$ is a directed graph $\vec{H}$ with the same vertex set, including all the arcs of $\vec{G}$ and such that, if (x, z) and (y, z) are arcs of $\vec{G}$, then (x, y) or (y, x) is an arc of $\vec{H}$.

A 1-*transitive fraternal augmentation* of $\vec{G}$ is a directed graph $\vec{H}$ with the same vertex set, including all the arcs of $\vec{G}$ and such that, for any vertices x, y, z,

- if (x, z) and (y, z) are arcs of $\vec{G}$ then (x, y) or (y, x) is an arc of $\vec{H}$ (*fraternity*),
- if (x, z) and (z, y) are arcs of $\vec{G}$ then (x, y) is an arc of $\vec{H}$ (*transitivity*).

A *fraternal augmentation* of a directed graph $\vec{G}$ is a sequence $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \vec{G}_{i+1} \subseteq \cdots$, such that $\vec{G}_{i+1}$ is a 1-fraternal augmentation of $\vec{G}_i$ for any $i \geq 1$. Of course this sequence of 1-fraternal augmentations eventually becomes constant, $G_i = G_{i+1}$, and this graph is called the *fraternal augmentation of* $\vec{G}$.

Similarly, a *transitive fraternal augmentation* of a directed graph $\vec{G}$ is a sequence $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \vec{G}_{i+1} \subseteq \cdots$ such that $\vec{G}_{i+1}$ is a 1-transitive fraternal augmentation of $\vec{G}_i$ for any $i \geq 1$. Again, this sequence of 1-transitive fraternal augmentations eventually becomes constant, $G_i = G_{i+1}$, and this graph is called the *transitive fraternal augmentation of* $\vec{G}$.

3. Fraternal augmentation and arrangeability

3.1. Acyclic fraternal augmentation

The link between arrangeability and fraternal augmentation will be made clear by the following theorem which states that p-arrangeable graphs are exactly those graphs admitting an acyclic 1-fraternal augmentation with *maximum indegree* $\Delta^- = p$:

Theorem 3.1. *Let G be a graph, p a positive integer. Then G is p-arrangeable if and only if it has an orientation $\vec{G}$ with an acyclic 1-fraternal augmentation $\vec{H}$ such that $\Delta^-(\vec{H}) \leq p$.*

Proof. By definition of a p-arrangeable graph, the vertices of G can be ordered as $v_1, v_2, \ldots, v_n$ such that $|N_{L_i}(N_{R_i}(v_i))| \leq p$ for each $1 \leq i \leq n$, where $L_i = \{v_1, v_2, \ldots, v_i\}$, $R_i = \{v_{i+1}, v_{i+2}, \ldots, v_n\}$. Then consider the orientation $\vec{G}$ of G such that any edge $\{v_i, v_j\}$ is oriented from v_i to v_j if $i < j$. Define a 1-fraternal orientation $\vec{H}$ of $\vec{G}$ as follows: for $1 \leq i < j < k \leq n$, if (v_i, v_k) and (v_j, v_k) are arcs of $\vec{G}$ then (v_i, v_j) is an arc of $\vec{H}$. Let $1 \leq i \leq n$. The number of arcs entering v_i in $\vec{H}$ is at most $|N_{L_i}(N_{R_i}(v_i))|$ and thus at most p; hence $\Delta^-(\vec{H}) \leq p$ and $\vec{H}$ is obviously acyclically oriented.

Conversely, assume that $\vec{G}$ is an orientation of G with an acyclic 1-fraternal augmentation $\vec{H}$. Consider any ordering $v_1, v_2, \ldots, v_n$ of the vertices of G which is compatible with the acyclic orientation of $\vec{H}$. Then for any vertex v_i of G, $|N_{L_i}(N_{R_i}(v_i))|$ is equal to the number of arcs entering v_i in $\vec{H}$; hence $|N_{L_i}(N_{R_i}(v_i))| \leq \Delta^-(\vec{H})$ and thus G is $\Delta^-(\vec{H})$-arrangeable. □

3.2. Transitive fraternal augmentation

Now we shall replace the acyclicity condition by a transitivity condition. In this context we have only the following:

Lemma 3.2. *Let p be an integer, let G be a graph, let $\vec{G}$ be an orientation of G and let $\vec{H}$ be a 1-transitive fraternal augmentation of G.*

Then G is $(\Delta^-(\vec{G}) + 2\nabla_0(H))$-admissible and thus $((\Delta^-(\vec{G}) + 2\nabla_0(H))^2 - \Delta^-(\vec{G}) - 2\nabla_0(H) + 1)$-arrangeable.

Proof. Let $p = \Delta^-(\vec{G}) + 2\nabla_0(H)$. Let $i \in \{0, 1, \ldots, n\}$ be the least integer such that there exist (distinct) vertices $v_{i+1}, v_{i+2}, \ldots, v_n$ with the following property: for all $j = i, i+1, \ldots, n$, G has no $V(G) - \{v_{j+1}, v_{j+2}, \ldots, v_n\}$-fan with center v_j of size $p+1$. If $i \neq 0$, let $M = V(G) - \{v_{i+1}, v_{i+2}, \ldots, v_n\}$. This set is non-empty and such that for every $v \in M$ there is an M-fan in G with center v of size $p+1$.

For $v \in M$, let $F(v)$ be an M-fan with center v with cardinality at least $p+1$. Associate a type with M-blades B in $F(v)$ as follows:

- type $= 1$ if $B = \{a\}$ is a singleton and $(a, v) \in E(\vec{G})$,
- type $= 2$ if $B = \{a\}$ is a singleton and $(v, a) \in E(\vec{G})$,
- type $= 3$ if $B = \{a, b\}$, $b \notin M$, $(b, v) \in E(\vec{G})$, $(b, a) \in E(\vec{G})$,
- type $= 4$ if $B = \{a, b\}$, $b \notin M$, $(b, v) \in E(\vec{G})$, $(a, b) \in E(\vec{G})$,

- type $= 5$ if $B = \{a, b\}$, $b \notin M$, $(v, b) \in E(\vec{G})$, $(b, a) \in E(\vec{G})$,
- type $= 6$ if $B = \{a, b\}$, $b \notin M$, $(v, b) \in E(\vec{G})$, $(a, b) \in E(\vec{G})$.

Then if B has type $1, 2, 4, 5, 6$ the vertices a and v are adjacent in H. Thus $\sum_{v\in M} |\{B \in F(v), \text{type}(v) \neq 3\}| \leq \sum_{v\in M} \mathrm{d}_H(v) \leq 2\nabla_0(H)|M|$. Now remark that two distinct M-blades of type 3 with center v use two different arcs entering v. As the maximum indegree of $\vec{G}$ is $\Delta^-(G)$, we get $\sum_{v\in M} |\{B \in F(v), \text{type}(v) = 3\}| \leq \sum_{v\in M} \mathrm{d}^-(v) \leq \Delta^-(G)|M|$. Altogether, as $|F(v)| \geq p + 1$ for any $v \in M$,

$$p + 1 \leq 2\nabla_0(H) + \Delta^-(G) \tag{2}$$

which contradicts the definition of p.

Hence $i = 0$, and $v_1, v_2, \ldots, v_n$ is an enumeration of the vertices of G showing that G is p-admissible. By Lemma 2.2, we deduce that G is $((\Delta^-(G) + 2\nabla_0(H))^2 - \Delta^-(G) - 2\nabla_0(H) + 1)$-arrangeable. □

4. Transitive fraternal augmentation and the grad

Lemma 4.1. *Let $\vec{G}$ be an acyclically oriented simple directed graph.*

Then there exists an edge coloring Υ using at most $2\Delta^-(\vec{G})$ colors such that any color induces a star forest oriented outward.

Proof. Let v be a sink. Color $G - v$ by induction with colors in $\{1, \ldots, 2\Delta^-(\vec{G})\}$. For each arc (x, v) entering v, at least $\Delta^-(G)$ colors among $\{1, \ldots, 2\Delta^-(\vec{G})\}$ are not present in an arc entering x. As there are at most $\Delta^-(\vec{G})$ arcs entering v, one can choose a suitable color for each arc entering v such that all these arcs get a different color. □

Lemma 4.2. *Let $\vec{G}$ be an acyclically directed graph with maximum indegree $\Delta^-(\vec{G})$. Then $\vec{G}$ has a 1-fraternal augmentation $\vec{H}$ such that $\Delta^-(\vec{H}) \leq \Delta^-(G)(1 + 2\nabla_1(G))$.*

Proof. Let Υ be the edge coloring defined in Lemma 4.1.

For any color α in $1, \ldots, \Upsilon(E(\vec{G}))$, let $\vec{G}_\alpha$ be the graph obtained from $\vec{G}$ by contracting all the edges of color α.

Let (x, z), (y, z) be arcs of $\vec{G}$ (x, y, z being distinct vertices) such that $\Upsilon((x, z)) = \alpha$. Then x and y are distinct and adjacent in G_α. As G_α may be oriented with indegree at most $\nabla_0(G_\alpha) \leq \nabla_1(G)$ we get that $\vec{G}$ has a 1-fraternal augmentation $\vec{H}$ with indegree bounded by $\Delta^-(\vec{H}) \leq \Delta^-(G) + |\Upsilon(E(\vec{G}))|\nabla_1(G)$. □

Corollary 4.3. *Every graph G has an acyclic orientation $\vec{G}$ with maximum indegree $2\nabla_0(G)$ and with a 1-transitive fraternal augmentation $\vec{H}$ such that $\Delta^-(\vec{H}) \leq 2\nabla_0(G)(1 + 2\nabla_1(G))$.*

Proof. It is a well known fact that G has an acyclic orientation with indegree at most $2\nabla_0(G)$. □

Theorem 4.4. *Every graph G is $(4\nabla_0(G)(\nabla_1(G) + \nabla_0(G) + 1))$-admissible and hence p-arrangeable for*

$$p = 16\nabla_0(G)^2\,(\nabla_1(G) + \nabla_0(G) + 1)^2 = O(\nabla_0(G)^2\nabla_1(G)^2).$$

Proof. This is a direct consequence of Corollary 4.3 and Lemma 3.2. □

We also list an application to the game chromatic number. This is an easy consequence of the following result:

Theorem 4.5 (*Kierstead, Trotter [11]*). *Let k and t be positive integers. If a k-admissible graph has chromatic number t, then its game chromatic number is at most $kt + 1$.*

Corollary 4.6. *Every graph G with $\nabla_0(G) \geq 1$ has game chromatic number at most $4\nabla_0(G)(2\nabla_0(G) + 1)(\nabla_1(G) + \nabla_0(G) + 1) + 1 = O(\nabla_0(G)^2\nabla_1(G))$.*

5. Proof that topologically closed classes have bounded expansion

We recall the following result of Komlós and Szemerédi [26]: If a simple graph on n vertices has at least $\frac{1}{2}p^2n$ edges, then it has a K_p-subdivision. Hence a graph G with no K_p-subdivision is such that $\nabla_0(G) < \frac{p^2}{2}$. Inequalities for the grads of further ranks are inductively deduced using the following lemma:

Lemma 5.1. *Let H be a minor of depth 1 of a graph G. Assume H includes a subdivision of $K_{p'}$. Then G includes a subdivision of K_p if $p' \geq 2p^2 - 6p + 8$.*

Proof. If $p = 1, 2$ or 3 the result is obvious as $p' \geq p$ and G will obviously include a vertex, an edge or a cycle (respectively). Thus we may assume $p \geq 4$ and hence $p' - p(p-1) \geq \max(p, (p-2)(p-3)+2)$.

By considering a subgraph of G if necessary, we may assume that $V(G)$ is partitioned into $A_1, \ldots, A_i, \ldots, A_{p'}, L_{1,1}, \ldots, L_{i,j}, \ldots, L_{p',p'}$ where:

- for $1 \leq i \leq p'$, $G[A_i]$ is a star (possibly reduced to a single vertex or a single edge);
- for $1 \leq i < j \leq p'$, there exist $v_{i,j} \in A_i$ and $v_{j,i} \in A_j$ such that $G[L_{i,j} \cup \{v_{i,j}, v_{j,i}\}]$ is a path with endpoint $v_{i,j}$ and $v_{j,i}$.

For the sake of simplicity, we define $L_{j,i} = L_{i,j}$ and $L_{i,i} = \emptyset$. For a subset Y of $\{1, \ldots, p\}$ we also define G_Y as the subgraph of G induced by $\bigcup_{i \in Y} A_i \cup \bigcup_{i,j \in Y} L_{i,j}$.

We first claim the following result: Let N be a positive integer and let X be a subset of $\{1, \ldots, p'\}$ of cardinality at least $\max(N, (N-2)(N-3)+2)$. Then there exists a subset $X' = \{k_{a,1}, \ldots, k_{a,N}\}$ of X of cardinality $(N-1)$ such that there exists in $G_{X'}$ a spider (that is: a subdivision of a star) with center $r_a \in A_{k_{a,a}}$ and leaves $l_{a,1}, \ldots, l_{a,a-1}, l_{a,a+1}, \ldots, l_{a,N}$ with $l_{a,i} \in L_{a,k_{a,i}}$. This claim is easily proved as follows: Assume that no vertex of $A_{k_{a,a}}$ has degree at least $(N-1)$ in G_X. Then $|X| - 1 \leq (N-2)(N-3)$, a contradiction. Choose for r_a any vertex of $A_{k_{a,a}}$ with degree at least $(N-1)$ in G_X. Then there exists in G_X a spider with center r_a and at least $(N-1)$ leaves belonging to different $A_{k_{a,i}}$.

Assume $p' - N(N-1) \geq (N-2)(N-3) + 2$, i.e. $p' \geq 2N^2 - 6N + 8$. Using the previous claim, we inductively define $Z_1, \ldots, Z_N$ with $Z_i = \{k_{i,1}, \ldots, k_{i,N}\}$ such that G_{Z_i} contains a spider with center $r_i \in A_{k_{i,i}}$ and leaves $l_{i,j} \in A_{k_{i,j}}$; to construct Z_i, we consider $X = \{1, \ldots, p'\} \setminus \bigcup_{1 \leq j < i} Z_j$. Then G includes a subdivision of K_N with principal vertices $r_1, \ldots, r_N$ as the union of all the spiders (and connections within the $L_{i,j}$ if necessary). □

Corollary 5.2. *Let G be a graph with no K_p-subdivision, let r be a positive integer. Then $\nabla_r(G) < 2^{r-1}p^{2^{r+1}}$.*

6. Concluding remarks

1. The principal result of [18] states that the bounded expansion is preserved under lexicographic product: For graphs G, G' the lexicographic product $G \bullet G'$ is the graph with vertices $V(G) \times V(G')$ and edges $((x, x'), (y, y'))$ where either $(x, y) \in E(G)$ or $x = y$ and $(x', y') \in E(G')$. (That is, we replace each vertex of G by a copy of G' and join these copies by completely according to edges of G.) We have the following (Lemma 5.2 of [18]):

There exists a polynomial $P(x, y)$

$$\nabla_1(G \bullet K_k) \leq P(k, \nabla_1(G)).$$

It follows that for any class $\mathcal{K}$ of graphs with bounded ∇_1 the class of graphs $\{G \bullet K_k; G \in \mathcal{K}\} = \mathcal{K} \bullet K_k$ has also a bounded ∇_1. Thus such classes have linear Ramsey number. We list some particular instances of this construction.

2. It is easy to see that any graph G has a subdivision G' such that $G' \in \mathcal{P} \bullet K_2$ where $\mathcal{P}$ is the class of planar graphs.

3. The preceding remark can be proved directly: For any graph G, if we replace each edge by a path of length ≥ 8 (i.e. if we subdivide each edge by seven vertices), then the resulting graph G' satisfies $\nabla_1(G') \leq 4$.(These two remarks relate to [1].)

4. If we replace each edge by a k-th power of a path of length ≥ 8 then we again obtain a graph with bounded ∇_1.

All the above classes have linear Ramsey number. They also have bounded game chromatic number.

It is not known whether the structure of graphs with unbounded ∇_1 and bounded ∇_0 could be characterized.

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